ESTIMATION OF POST-PEAK BEHAVIOUR OF BRITTLE ROCKS USING A CONSTITUTIVE MODEL FOR ROCK JOINTS

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ABSTRACT: The evaluation of rockburst potential near underground structures requires the knowledge of the post-peak behaviour of rock and rock masses. However, the laboratory determination of the post-peak properties during uniaxial compression tests of brittle rocks is often difficult to realise. This paper shows how a constitutive model recently developed for rock joints can be used to estimate this post-peak behaviour. The proposed approach leads to this estimation when full post-peak curves have been obtained from triaxial compression tests or from uniaxial compression tests on non-standard shape samples. After presenting the approach, a validation is performed using results taken from the literature.

1. INTRODUCTION

In many Canadian underground mines, there is a strong concern for the possible occurrence of rockbursts. Several authors have proposed that the violent nature of pillar bursts can be explained by the relative difference between the post-peak stiffness of the pillar and the stiffness of the surrounding rock mass loading the pillar (e.g. Cook 1965; Starfield and Fairhurst 1968; Gill et al. 1993). In the evaluation of the rockburst potential for this kind of rockbursts near underground openings, the determination of the post-peak behaviour of rocks is required to assess if a possible failure will be gradual or violent. However, the characteristics of the post-peak behaviour of hard rocks are usually extremely difficult to obtain in laboratory, even with very stiff or servo-controlled testing machines. Hence, in many cases, one must use indirect approaches to estimate this post-peak behaviour (Brady and Brown 1981; Aubertin et al. 1994; Joseph and Barron 2003).

Hard rocks in compression tests can go from a brittle behaviour to a semi-brittle behaviour when the height/width ratio of the sample is reduced or when the confining pressure is increased. In this case, failure seems to occur following the creation of a single shear fracture. When the peak strength is reached, the slope of the stress-strain curve is then controlled by the shear and normal deformation on the induced fracture(s) and the deformation of the intact rock matrix. In this paper, the authors present an analytical approach, based on a recently developed constitutive model for rock joint, to estimate the post-peak behaviour of rock samples submitted to uniaxial and triaxial compression tests. It is shown how the uniaxial post-peak behaviour can be estimated when results are available for the full post-peak stress-strain curve of triaxial tests or uniaxial tests on low height/width ratio samples. The approach is then applied on representative test results taken from the literature.

2. THE CSDS CONSTITUTIVE MODEL FOR ROCK JOINTS

A constitutive model, called the CSDS model (for Complete Stress-Displacement Surface), has been recently developed for rock joints by the authors (Simon 1999; Simon et al. 1999). The model is composed of two basic sets of equations. The first set defines the shear stress - shear displacement relationship while the second set relates the normal displacement to the shear displacement.

2.1 Shear stress-displacement relationship

The \( \tau = F(u) \) relationship is defined by:

\[
\tau = a + b \exp(-c u) - d \exp(-e u) \tag{1}
\]

where \( \tau \) is the shear stress (MPa), \( u \) is the shear displacement (mm), and \( a \) to \( e \) are model parameters that must satisfy the imposed conditions: \( c < e \), and \( a,b,c,d,e > 0 \). These parameter values can be determined from the following equations (Simon 1999, Simon et al. 1999):
\[ a = \tau_r \]  
\[ b = d - a \]  
\[ c = 5/u_r \]  
\[ \frac{d}{u_r} - \frac{u_r}{e} = \exp \left[ \frac{u_p}{5} \left( e - \frac{5}{u_r} \right) \right] = 0 \]  
\[ \frac{\tau_p - \tau_r}{d} \left( 1 - \exp \left( -\frac{5u_p}{u_r} \right) \right) = 0 \]  
\[ \exp \left( -\frac{5u_p}{u_r} \right) - \exp \left( -eu_p \right) \]

In these equations, \( \tau_r \) is the residual strength, \( \tau_p \) is the peak strength, \( u_p \) is the peak displacement and \( u_r \) is the displacement at the onset of \( \tau_r \). Here, \( u_p \) and \( u_r \) are considered material constants. Two non-linear equations (Eq. 5-6) must be solved simultaneously to evaluate the values of parameters \( d \) and \( e \) from a given set of data. These can be solved by standard iterative methods. Proper care must be taken when solving Equation 5 because it has two roots for \( e \). The larger value is used in order to satisfy the imposed condition \( c < e \). More details on the development of these equations can be found in Simon (1999) and Simon et al. (1999).

To evaluate the residual shear strength, a Coulomb criterion without cohesion can be used:

\[ \tau_r = \sigma_n \tan \phi_r \]

where \( \phi_r \) is the residual friction angle on the joint surface and \( \sigma_n \) is the normal stress acting on the rock joint. The peak shear strength \( \tau_p \) can be obtained using an appropriate existing peak shear strength criterion. The authors have used the LADAR criterion (Ladanyi & Archambault 1970) as modified by Saeb (1990). The peak shear strength is then given by:

\[ \tau_p = \sigma_n (l - a_i) \tan (i + \phi_r) + a_s \sigma_n \]

where (Ladanyi & Archambault 1970):

\[ i = \tan^{-1} \left[ \left( 1 - \frac{\sigma_n}{\sigma_T} \right)^{k_2} \tan i_0 \right] \]

\[ a_s = 1 - \left( \frac{\sigma_n}{\sigma_T} \right)^{k_1} \]

Saeb (1990) has further proposed the use of a Mohr-Coulomb strength criterion for the shearing of rock asperities:

\[ S_r = S_0 + \sigma_n \tan \phi_0 \]

In these equations, \( i_0 \) is the initial angle of asperities, \( \sigma_T \) is a transitional stress (taken as the uniaxial compressive strength, as suggested by Goodman 1976), \( k_1 \) and \( k_2 \) are material constants for the LADAR model (which take empirical values of 1.5 and 4 respectively), \( S_0 \) is the rock cohesion and \( \phi_0 \) is the rock friction angle.

2.2 Normal – shear displacements relationship

To describe the normal displacement \( (v) \) - shear displacement \( (u) \) relationship, an exponential formulation has also been used. This relation can be given in the form (Simon 1999, Simon et al. 1999):

\[ v = \beta_1 - \beta_2 \exp(-\beta_3 u) \]

where \( \beta_1 \), \( \beta_2 \) and \( \beta_3 \) are model parameters. The value of these parameters are given by:

\[ \beta_1 = u_r \left( \frac{\sigma_n}{\sigma_T} \right)^{k_2} \tan i_0 + \frac{\sigma_n}{k_m} V_m - \sigma_n \]

\[ \beta_2 = \beta_3 \frac{\sigma_n}{k_m V_m - \sigma_n} \]

\[ \beta_3 = \frac{l}{u_r} \]

where joint opening and compressive stress are positive; \( v \) is the normal displacement; \( V_m \) is the maximum closure (usually smaller than initial starting aperture); \( k_m \) is the initial normal stiffness of the joint; \( \sigma_n \) is the normal stress. The values of \( i_0 \) and \( k_2 \) are the same as in the LADAR peak shear strength criterion.

It was shown in previous publications (Simon 1999; Simon et al 1999) that the CSDS model can reproduce very closely the behaviour of rock joints, including the post-peak region. It is shown below that it can also be used to describe the post-failure behaviour of intact rock.
3. ESTIMATION OF POST-PEAK BEHAVIOUR OF BRITTLE ROCKS

In the following, the behaviour of brittle rocks in compression is briefly described. A more complete description of brittle rock behaviour can be found in Aubertin et al. (1998, 2000). Figure 1 shows a typical stress-strain curve of a brittle rock tested in compression. Such schematic diagram allows the identification of the different deformation stages. Stage I is associated to closure of microcracks and it reflects of the initial state of existing flaws in the material. Stage II is a linear elastic phase.

The onset of inelastic straining occurs at the beginning of stage III at point B. During stage III, there is some microcrack initiation, but little propagation, at least for the loading rates used in these tests. The cracks size appears to be limited to about the grain dimension (e.g. Huang et al, 1993), with an orientation nearly parallel to the major principal stress \( \sigma_1 \).

When the loading stress is increased further, stage IV is reached (point C) for a stress typically between 50 to 90% of the ultimate strength. At this point, the stress-axial strain curve clearly departs from linearity, and the lateral strain increases much more rapidly. The sample usually shows, around the same stress level, a volumetric strain reversal, from a compressive to a relatively dilatant behaviour. At the beginning of stage IV, some sliding along crack faces is induced, which in turns allows for the appearance of longer cracks following coalescence of existing flaws. As in stage III, however, it seems that crack distribution initially remains fairly uniform in the sample (Huang et al, 1993). It is only when the stress on the sample approaches very near the peak strength at point D, that larger cracks (and strain localisation) are observed (e.g., Wawersik et al, 1990).

The peak strength represents the maximum stress that a sample can support under a given set of loading conditions. In the post-peak stage, the rock sample becomes discontinuous, and its response may depend upon properties of the loading system as well as rock properties (Hakami 1988). It is often very difficult to obtain the post-peak curve of brittles rocks showing a very steep post-peak slope, as the strain energy contained in the system is higher than the work that the sample can do in this post-peak phase. This results in a violent failure of the sample with a complete loss of cohesion. In this case, the stress-strain curve cannot be followed in the post-peak regime.

3.1 Stress-strain analysis in post-peak phase

Figure 2 shows an idealized shear failure plane in a sample produced by a triaxial compression test. The stress conditions acting on this plane are given by:

\[
\sigma_n = \frac{1}{2}(\sigma_1 + \sigma_3) - \frac{1}{2}(\sigma_1 - \sigma_3) \cos 2\beta \quad [16]
\]

\[
\tau = \frac{1}{2}(\sigma_1 - \sigma_3) \sin 2\beta \quad [17]
\]

where \( \sigma_n \) and \( \tau \) are respectively the normal and the shear stresses, \( \sigma_1 \) is the major principal stress, \( \sigma_3 \) is the minor principal stress which equals the confining stress (with \( \sigma_2 = \sigma_3 \)), and \( \beta \) is the angle between the shear plane and the sample axis.

Once the shear plane is created around the peak strength, the variation of the axial strain will depend of the following displacements:

- the shear displacement \( u \) along the plane;
- the normal displacement (dilatancy) \( v \) caused by the shear displacement;
- the response of the rock matrix (taken as elastic) due to the axial stress drop.

Figure 1. The deformation stages of a rock specimen tested in compression (adapted from Paterson, 1978).

Figure 2. Single shear failure plane in a sample submitted to a triaxial compression test.
The variation of the axial strain caused by the shear displacement $\Delta e_s$ (compression is positive) is given by:

$$\Delta e_s = \frac{Lu \cos \beta}{L}$$ \hspace{1cm} [18]

The variation of the axial strain caused by the normal displacement along the plane $\Delta e_n$ (opening of the plane is positive) is given by:

$$\Delta e_n = -\frac{Lv \sin \beta}{L}$$ \hspace{1cm} [19]

The variation of the axial strain $\Delta e_e$ caused by the reduction of the axial stress is given by:

$$\Delta e_e = \frac{\Delta \sigma_1}{E}$$ \hspace{1cm} [20]

In these equations, $L$ is the initial length of the sample and $E$ is the elastic modulus of the rock matrix. Thus, the total axial strain $\varepsilon$ in the post-peak stage is given by:

$$\varepsilon = \varepsilon_p + \Delta e_s + \Delta e_n + \Delta e_e$$

$$= \varepsilon_p + \frac{Lu \cos \beta}{L} - \frac{Lv \sin \beta}{L} + \frac{\Delta \sigma_1}{E}$$ \hspace{1cm} [21]

where $\varepsilon_p$ is the axial strain at peak strength.

### 3.2 Determination of shear stress-displacement curve from a triaxial compression test curve

The first step in the estimation of the post-peak behaviour is to plot the shear stress-shear displacement curve from a complete triaxial compression test curve (Fig. 3). To do so, the following procedure is proposed:

- For each data point on the post-peak strain curve (Fig. 3), subtract the restored elastic strain (Eq. 20) and the axial peak strain $\varepsilon_p$. The difference becomes the strain caused by the shear and normal displacement on the failure surface.

- Since the normal displacement $v$ is a function of the shear displacement $u$ (Eq. 12), an iterative process must be performed to isolate the normal component. To do so, the suggested approach is to set $v=0$ and then calculate an initial value for $u$ from Eq. 21 (for a given or estimated angle $\beta$). With this initial value, calculate $v$ from Eq. 12 with the starting values of the CSDS model parameters. Only the variation of $v$ needs to be taken into account, so Eq. 13 can be modified so that $v(u=0) = 0$:

$$\beta_i = u_i \left[ 1 - \frac{\sigma_n}{\sigma_T} \right] \tan i_0$$ \hspace{1cm} [22]

The process is repeated until the value obtained for $u$ is constant. Because the normal stress acting on the failure surface is relatively high, the dilatancy is fairly small, and only a few iterations are required (usually less than five). The results can then be plotted, as shown in Figure 4.

- From this graph, the CSDS model parameters can be adjusted to fit the experimental curve, as shown in Figure 4. This adjustment is more representative when several testing curves are available for different levels of confining stress.

![Figure 3. Example of a typical stress-strain curve obtained from a triaxial compression test.](image3)

![Figure 4. Example of a shear stress-shear displacement graph obtained with the proposed approach.](image4)
3.3 Estimation of post-peak behaviour

Once the material properties used in the CSDS model parameters have been determined from the triaxial compression test curves, the process can be performed backward to plot the axial stress-axial strain curve.

To plot the post-peak axial curve, the following steps can be used:

- For decreasing values of \( \sigma_1 \) (with \( \sigma_3 \) = constant), calculate the shear and normal stress acting on the failure plane (Eq. 16-17).
- For the corresponding values of \( \tau \) and \( \sigma_n \), calculate the ensuing CSDS model parameters \( a \) to \( e \), \( \beta_1 \), \( \beta_4 \) and \( \beta_5 \) (Eq. 2-15).
- Calculate the corresponding \( u \) values (Eq. 1). It should be mentioned that it is not possible to isolate \( u = f(\tau) \) in equation 1 since two values are possible, one before the peak strength and one for the post-peak phase. The post-peak value (which leads to a reduction of strength) is the value needed. A spreadsheet solver can easily perform this task with the imposed condition \( u > u_p \).
- From the \( u \) values obtained, calculate the normal displacement \( v \) (Eq. 12) with \( \beta_1 \) given by Eq. 22.
- Then calculate the axial strain for representative values of \( \sigma_1 \) with Eq. 21.

Figure 5 shows a sample result obtained with this approach. It is seen that the modelled curve can approximate well this idealised post-peak stress-strain curve.

![Figure 5](image)

Figure 5. Example of a post-peak curve obtained with the CSDS model.

4. SAMPLE APPLICATION

To illustrate how the proposed approach can be implemented with actual test results, two sample applications are presented using experimental curves taken from the literature. One case shows triaxial compression tests with different confining pressures (from 0 to 14 MPa), and the other one included uniaxial compression tests performed on samples with different height to width ratios.

4.1 Application with triaxial compression tests

In this first example, uniaxial and triaxial compression tests on sandstone, performed by Price (1979), are analysed. To validate (at least partially) the approach, the CSDS model parameters are calibrated on the triaxial tests and the post-peak behaviour of the uniaxial test is estimated with the same parameters values. Figure 6 shows the curves used for the analysis and the results of the proposed approach. Table 1 shows the values of the CSDS model that were determined for each test.

The value of \( \sigma_T \) is taken as the uniaxial compressive strength, which is 95 MPa. The value of \( S_0 \) and \( \phi_0 \) are obtained by using a Mohr-Coulomb failure criterion for the given peak strength results. The value of \( \phi_1 \) is obtained by using a Mohr-Coulomb without cohesion on the residual strength obtained from the different curves. The elastic moduli \( E \) are taken at 50% of the peak strength. The values of the other CSDS parameters were adjusted so the model could fit the \( \tau-u \) curve for the three available triaxial tests as shown in Figure 7. The same values were then used for the uniaxial compression test. The \( \beta \) angles were determined so the peak shear stress given by Eq. 17 would equals the peak shear strength given by Eq. 8.

As can be seen in these figures, the approach shows a good correlation with the experimental curves. This includes the uniaxial test result, which in a sense has been "predicted" from the curves obtained under triaxial compression, following the approach described above. Hence, modelling with CSDS allows the user to estimate the more problematic post-peak uniaxial curve from a few triaxial tests results.

![Figure 6](image)

Figure 6. Uniaxial and triaxial compression test results on 75 mm diameter sandstone with a height/width ratio of 2 (data from Price 1979).
The same approach can also be used with uniaxial test results on samples with different height/width ratios. When decreasing the height/width ratio of the samples tested, rock material can go from a brittle behaviour to a semi-brittle behaviour. Then, it is possible to get a full post-peak curve of a brittle rock by reducing its height/width ratio and then, to estimate the post-peak behaviour of a standard shape sample. Figure 8 shows uniaxial compression test results on 102 mm diameter samples of different shapes of Georgia Cherokee Marble. Table 2 shows the parameters values used. The value of $\sigma_T$ is taken as the uniaxial compressive strength for a L/D ratio of 2, which is 85 MPa. The elastic moduli $E$ are taken at 50% of the peak strength. The values of the other CSDS parameters were adjusted so the model could fit the $\tau$-$u$ curve for the L/D ratio of 1 as shown in Figure 9. The $\beta$ angles were adjusted so the resulting peak shear stress would equal the peak shear strength given by the LADAR model. Here again, the proposed approach shows a good correlation with the experimental curves.

5. DISCUSSION

The major difficulty in using the proposed approach resides in the determination of the different parameters of the CSDS model and of the failure plane angle. The significance and influence of these parameters are discussed in the following.

5.1 Angle of the failure plane $\beta$

This is one of the key parameter in the proposed approach. This angle defines the normal and shear stresses acting on the failure plane. This angle can usually be obtained by a visual inspection of the sample after the test. For uniaxial compression tests of brittle rocks, where the failure of the sample may be sudden, it may not be possible to assess this angle and other means must be used. This is the case for the sample
applications presented here where no information on the angle values was available.

Figure 8. Uniaxial compression test results on a Georgia Cherokee marble with different height/width (L/D) ratios (after Hudson et al. 1971).

Figure 9. Shear stress-strain curves obtained with the proposed approach for Fig. 8 and application of the CSDS model

When the angle of the failure plane cannot be obtained from visual inspection, it can be estimated from the peak shear strength/normal stress ratio \( \tau_s/\sigma_n \). As the value of \( \beta \) increases, this ratio will decrease. Although this ratio is not a constant in the LADAR criterion (for a given set of material properties), at high normal stress (or at high values of \( \sigma_n/\sigma_T \)), the ratio shows very little variation. For the uniaxial and triaxial compression tests shown here, the normal stress acting on the failure surface is high enough that the ratio \( \tau_s/\sigma_n \) can be taken as a material constant. In the examples presented above, the angles were assessed so the peak shear stress would equals the peak shear strength given by the LADAR model (Eq. 8).

One of the assumptions made with the proposed approach is that failure follows the creation of a distinct single shear plane. The \( \beta \) value must then be at least higher than \( \tan^{-1}(D/L) \). This leads to a minimum value of \( \beta \) between 21-27° for standard L/D ratios between 2-2.5. These angles are in agreement with several experimental results (e.g. Mogi 1966; Hakami 1988). Typical values for \( \beta \) range between 20° and 40° (Hakami 1988).

However, several authors (e.g. Hawkes and Mellor 1970; Hakami 1988) have shown that other modes of failure can occur and that fractures can also form at the ends of the sample. In this case, the proposed approach would not be strictly valid. This could explain why the correlation for the L/D value of 1/2 in Fig. 9 is not as good. Moreover, the authors were not able to reproduce post-peak behaviour of a sample with a L/D ratio of 1/3 (not shown), as the behaviour of the sample appears almost ductile (with no reduction of strength). In this case however, the influence of platen friction may become so important that the stress state in the sample has little to do with the expected (or idealised) one.

5.2 Shear displacement parameters

The shear displacement has a great influence on the post-peak behaviour. When analysing the different components of the total axial strain defined in the proposed approach (Figure 10), it is observed that the shear displacement represents about three times the other components (elastic and normal strain) of the total post-peak strain. An accurate description of the shear displacement is then essential. As shown in previous publications (Simon 1999; Simon et al. 1999) and in Figure 7 and 9, the formulation used in the CSDS model is very effective in describing the post-peak behaviour of rock joints.

In the determination of the CSDS model parameters, the value of the peak shear displacement \( u_p \) was almost 0. However, this does not reflect typical values obtained from standard direct shear test on rock joints. In the proposed approach, only the post-peak behaviour is analysed, starting from the peak value of the decreasing shear stress (as shown in Fig. 4). The value of \( u_p \) should then be set to zero, but this is not possible with the CSDS model formulation, as it would lead to a division by zero in Eq.6. This would also indicate that the failure plane is progressively created even before the peak axial strength is reached, and that the mobilization of the shear strength on the failure plane could contributes to the maximum strength attained (and thus, to the pre-peak deformation).

Hence, a more complete and comprehensive analysis would involve looking at strains before the peak at the onset of localisation (somewhere between point C and D on Fig. 1). However it is certainly not an easy task to determine the onset condition for the creation of the failure plane. This uncertainty may also help explain the discrepancy of the model with experimental curves just near the peak strength, as illustrated with the curve for \( \sigma_s = 14 \) MPa in Fig. 7 and the curves for L/D = 1/2 and L/D = 3 in Fig. 9. Nonetheless, even if strains do not exactly match each other at peak strength, the slopes of the post-peak curves are very similar. This would be very useful for many applications, such as the analysis of rockburst potential, where the value of the post-peak slope is of interest.
Figure 10. Typical components of the total post-peak strain: shear displacement (Eq. 18), normal displacement (Eq. 19) and elastic (Eq. 20).

Also, several parameters need to be evaluated to apply the CSDS model, but the information that can be extracted from axial stress-strain curves is limited. This leads to some uncertainties in the values of the different parameters. Triaxial tests at different levels of confinement lead to a better evaluation of the required parameters than tests on different samples with different height/width ratios for instance. From the triaxial tests, one can directly determine the Mohr-Coulomb strength parameters ($S_0$, $\phi_0$ and $\phi_r$) used in the LADAR peak shear strength criterion for rock joints and for the rock residual strength.

Like any other testing program on rock material, the values obtained can show great variation from one sample to the other. Then, a minimum number of tests should be performed to insure some statistical validity.

5.3 Normal displacement parameters

In the proposed approach, the information that can be drawn from the axial stress-strain curves to estimate the normal displacement parameters of the CSDS model is almost inexistent. It then becomes difficult to estimate plausible values for these parameters. Nonetheless, it can be shown that these uncertainties have a limited effect on the results. The parameters that are used to define the normal behaviour are $\sigma_T$, $u_i$, $k_n$, $V_m$, and $i_0$. The value of $\sigma_T$, if taken as the uniaxial compression strength, is typically known (with the usual uncertainty). The value of $u_i$ can be assessed from the shear stress-displacement curves (Fig 7 & 9). Because the normal stress acting on the failure plane is relatively high during these tests near the peak strength, it could be shown that the influence of parameters $k_n$ (the initial normal stiffness) and $V_m$ (maximum normal closure) is negligible.

The most significant parameter for evaluating the joint normal behaviour is then the initial angle of asperities $i_0$. As the value of $i_0$ increases, the normal displacement tends to increase. However, the value of $i_0$ also influences the peak strength of the failure plane given by the LADAR criterion (Eq. 8-11). Thus, its value can be estimated from the shear displacement curve.

6. CONCLUSION

In this paper, the authors have proposed a new approach to describe, and in some case predict the post-peak behaviour of brittle rocks samples submitted to uniaxial and triaxial compression tests. The proposed approach is based on the use of a recently developed rock joint constitutive model, called CSDS. Sample applications were presented and showed a good correlation with test results.

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